Chapter 11

Lecture: Spherical Black Holes

One of the most spectacular consequences of general relativity is the prediction that gravitational fields can become so strong that they can effectively trap even light.

- Space becomes so curved that there are no paths for light to follow from an interior to exterior region.
- Such objects are called black holes, and there is extremely strong circumstantial evidence that they exist.
- In this chapter we apply the Einstein theory of gravity to the idea of black holes using the Schwarzschild solution.
- In the next chapter we shall take a first step in considering how gravitational physics is altered if the principles of quantum mechanics come into play (Hawking black holes),
- In the chapter after that we shall consider how the Schwarzschild solution is modified if a black hole is assumed to possess angular momentum (Kerr black holes.)

11.1 Schwarzschild Black Holes

There is an event horizon in the Schwarzschild spacetime at $r_S = 2M$, which implies a black hole inside the event horizon (escape velocity exceeds c).

Place analysis on firmer ground by considering a spacecraft approaching the event horizon in free fall (engines off).

- For simplicity, we assume the trajectory to be radial.
- Consider from two points of view
  1. From a very distant point at constant distance from the black hole (professors with martinis).
  2. From a point inside the spacecraft (students).
- Use the Schwarzschild solution (metric) for analysis.

11.1.1 Approaching the Event Horizon: Outside View

We consider only radial motion. Setting $d\theta = d\varphi = 0$ in the line element

$$\begin{aligned} ds^2 = -d\tau^2 &= -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ &= -\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2. \end{aligned}$$

- As the spacecraft approaches the event horizon its velocity as viewed from the outside in a fixed frame is $v = dr/dt$.
- Light signals from spacecraft travel on the light cone ($ds^2 = 0$) and thus from the line element

$$v = \frac{dr}{dt} = \left(1 - \frac{r_S}{r}\right).$$

- As viewed from a distance outside r_S, *the spacecraft appears to slow as it approaches* r_S *and eventually stops as* $r \to r_S$.
- Thus, from the exterior we would never see the spacecraft cross r_S: *its image would remain frozen at* $r = r_S$ *for all eternity.*

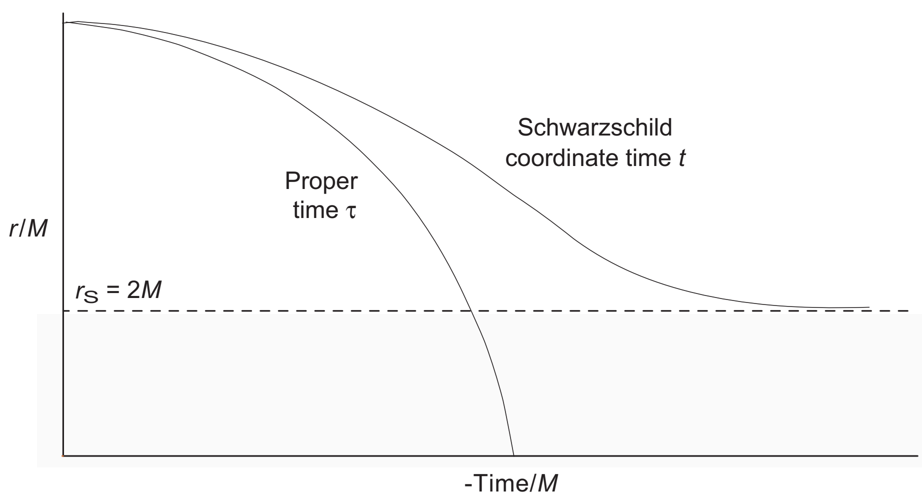
But let us examine what this means a little more carefully. Rewrite

$$\frac{dr}{dt} = \left(1 - \frac{r_S}{r}\right) \quad \longrightarrow \quad dt = \frac{dr}{1 - r_S/r}.$$

- As $r \longrightarrow r_S$ time between successive wave crests for the light wave coming from the spacecraft tends to infinity and therefore

$$\lambda \longrightarrow \infty \qquad \nu \longrightarrow 0 \qquad E \longrightarrow 0.$$

- The external observer not only sees the spacecraft slow rapidly as it approaches r_S, but the spacecraft image is observed to *strongly redshift* at the same time.
- This behavior is just that of the coordinate time already seen for a test particle in radial free fall:



- Therefore, more properly, the external observation is that *the spacecraft approaching r_S rapidly slows and redshifts until the image fades from view before the spacecraft reaches r_S.*

11.1.2 Approaching the Event Horizon: Spacecraft View

Things are very different as viewed by the (doomed) students from the interior of the spacecraft.

- The occupants will use their own clocks (*measuring proper time*) to gauge the passage of time.
- Starting from a radial position r_0 outside the event horizon, the spacecraft will reach the origin in a proper time

$$\tau = -\frac{2}{3}\frac{r_0^{3/2}}{(2M)^{1/2}}.$$

- The spacecraft occupants will generally notice no spacetime singularity at the horizon.
- Any tidal forces at the horizon may be very large but will remain finite (Riemann curvature is finite at the Schwarzschild radius).
- The spacecraft crosses r_S and reaches the (real) singularity at $r = 0$ in a finite amount of time, where it would encounter infinite tidal forces (Riemann curvature has components that become infinite at the origin).
- The trip from r_S to the singularity is very fast (Exercise):
  1. Of order 10^{-4} seconds for stellar-mass black holes.
  2. Of order 10 minutes for a billion solar mass black hole.

11.1.3 Lightcone Description of a Trip to a Black Hole

It is highly instructive to consider a lightcone description of a trip into a Schwarzschild black hole.

- Assuming radial light rays

$$d\theta = d\varphi = ds^2 = 0$$

  the line element reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = 0,$$

- Thus the equation for the lightcone at some local coordinate r in the Schwarzschild metric can be read directly from the metric

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}.$$

  where

  - The plus sign corresponds to outgoing photons (r increasing with time for $r > 2M$)
  - The minus sign to ingoing photons (r decreasing with time for $r > 2M$)

- For large r

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}.$$

  becomes equal to ± 1, as for flat spacetime.

- However as $r \to r_S$ the forward lightcone opening angle tends to zero as $dt/dr \to \infty$.

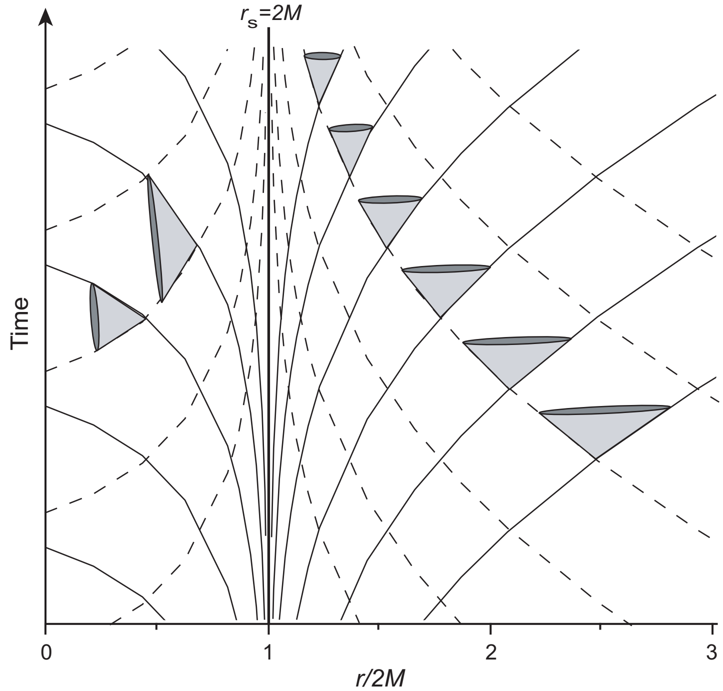


Figure 11.1: Photon paths and lightcone structure of the Schwarzschild spacetime.

Integrating

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}.$$

gives

$$t = \begin{cases} -r - 2M \ln|r-1| + \text{ constant} & \text{(Ingoing)} \\ r + 2M \ln|r-1| + \text{ constant} & \text{(Outgoing)} \end{cases}$$

- The null geodesics defined by this expression are plotted in Fig. 11.1.
- The tangents at the intersections of the dashed and solid lines define local lightcones corresponding to dt/dr, which are sketched at some representative spacetime points.

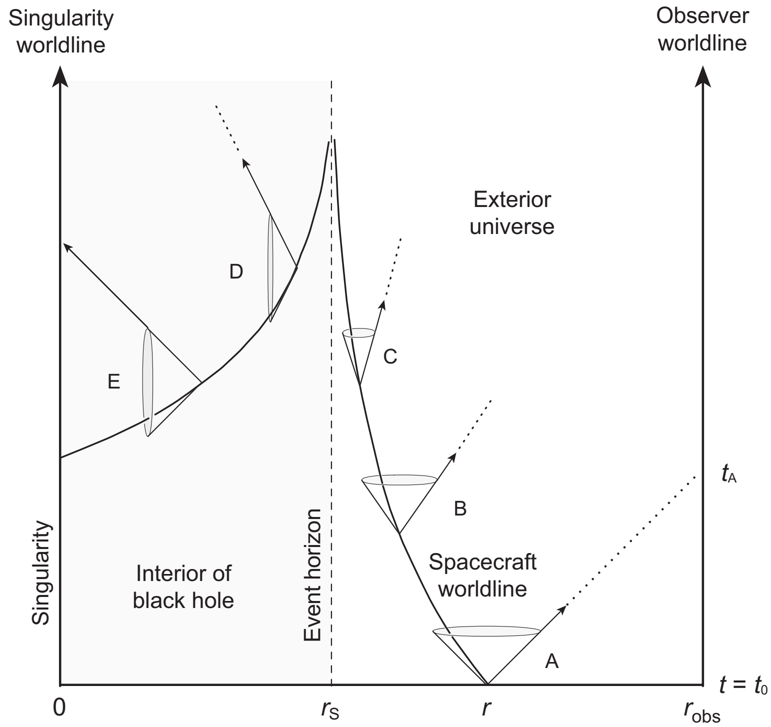
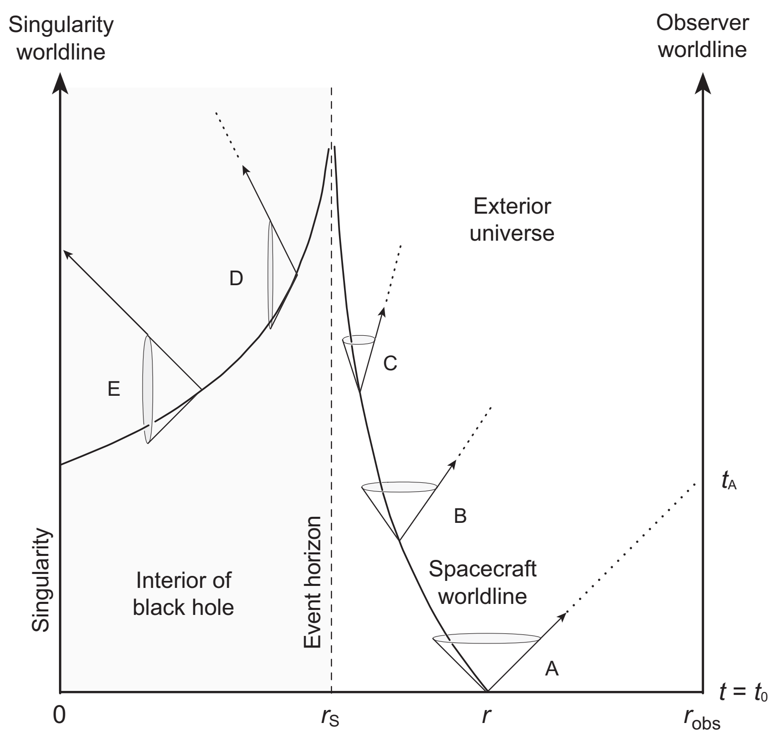


Figure 11.2: Light cone description of a trip into a Schwarzschild black hole.

- The worldline of a spacecraft is illustrated in Fig. 11.2, starting well exterior to the black hole. The gravitational field there is weak and the light cone has the usual symmetric appearance.
- As illustrated by the dotted line from A, a light signal emitted from the spacecraft can intersect the worldline of an observer remaining at constant distance $r_{\rm obs}$ at a finite time $t_{\rm A} > t_0$.
- As the spacecraft falls toward the black hole on the worldline indicated the forward light cone begins to narrow since

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}.$$



- Now, at B a light signal can intersect the external observer worldline only at a distant point in the future (arrow on light cone B).
- As the spacecraft approaches r_S, the opening angle of the forward light cone tends to zero and a signal emitted from the spacecraft tends toward *infinite time* to reach the external observer's worldline at r_{obs} (arrow on light cone C). *The external observer sees infinite redshift.*

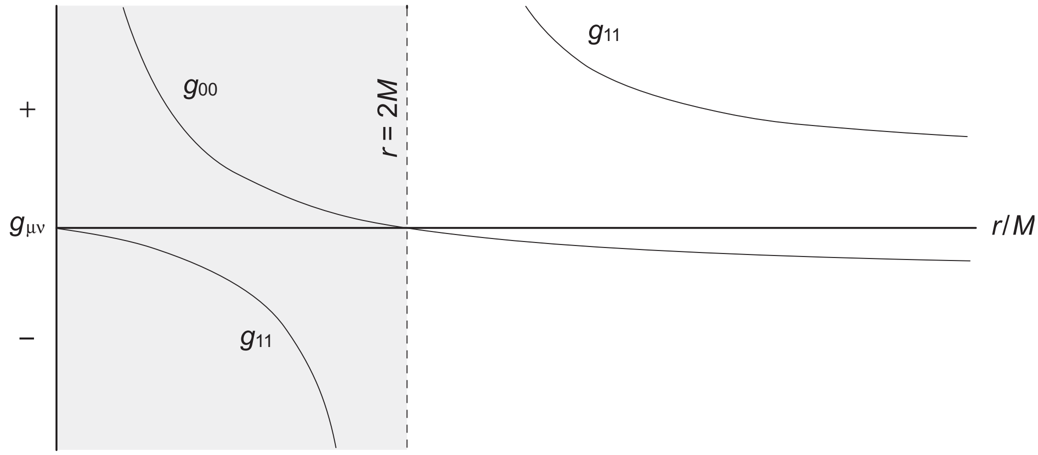


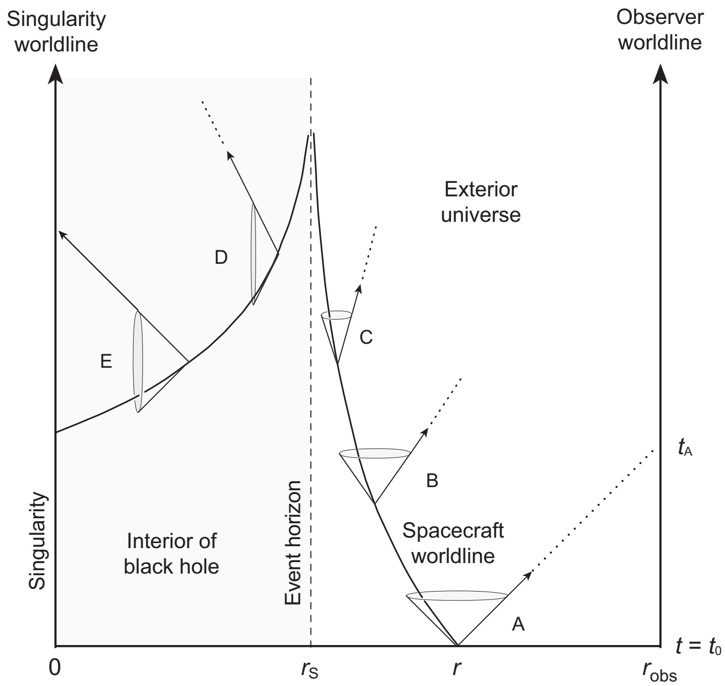
Figure 11.3: Spacelike and timelike regions for g_{00} and g_{11} in the Schwarzschild metric.

Now consider light cones interior to the event horizon.

- From the structure of the radial and time parts of the Schwarzschild metric illustrated in Fig. 11.3, we observe that dr and dt reverse their character at the horizon ($r = 2M$) because the metric coefficients g_{00} and g_{11} *switch signs* at that point.

  1. Outside the event horizon the t direction, $\partial/\partial t$, is timelike ($g_{00} < 0$) and the r direction, $\partial/\partial r$, is spacelike ($g_{11} > 0$).
  2. Inside the event horizon, $\partial/\partial t$ is spacelike ($g_{00} > 0$) and $\partial/\partial r$ is timelike ($g_{11} < 0$).

- Thus inside the event horizon the lightcones get rotated by $\pi/2$ relative to outside (space $\leftrightarrow$ time).

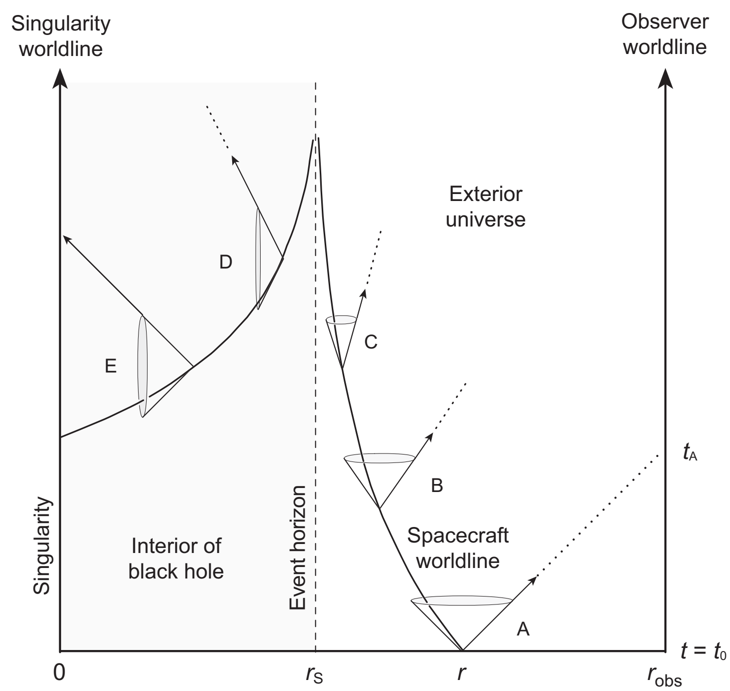


- The worldline of the spacecraft descends inside r_S because the coordinate time decreases (it is now behaving like r) and the decrease in r represents the passage of time, but the *proper time* is continuously increasing in this region.

- Outside the horizon r is a spacelike coordinate and application of enough rocket power can reverse the infall and make r begin to increase.

- Inside the horizon r is a timelike coordinate and no application of rocket power can reverse the direction of time.

- Thus, the radial coordinate of the spacecraft must decrease once inside the horizon, for the same reason that time flows into the future in normal experience (whatever that reason is!).



- Inside the horizon there are no paths in the forward light cone of the spacecraft that can reach the external observer at r_0 (the right vertical axis)—see the light cones labeled D and E.

> All timelike and null paths inside the horizon are bounded by the horizon and must encounter the singularity at $r = 0$.

- This illustrates succinctly the real reason that nothing can escape the interior of a black hole. *Dynamics (building a better rocket) are irrelevant:* once inside r_s the geometry of spacetime permits no forward light cones that intersect exterior regions, and no forward light cones that can avoid the origin.

Thus, there is no escape from the classical Schwarzschild black hole once inside the event horizon because

1. There are literally no paths in spacetime that go from the interior to the exterior.
2. All timelike or null paths within the horizon lead to the singularity at $r = 0$.

> "You can check out any time you want,
> But you can never leave!"
>
> *Hotel California*
> The Eagles

But notice the adjective "classical" . . . More later.

11.1.4 Eddington–Finkelstein Coordinates

The preceding discussion is illuminating but the interpretation of the results is complicated by the behavior near the coordinate singularity at $r = 2M$.

- In this section and the next we discuss two alternative coordinate systems that remove the coordinate singularity at the horizon.
- Although these coordinate systems have advantages for interpreting the interior behavior of the Schwarzschild geometry, the standard coordinates remain useful for describing the exterior behavior because of their simple asymptotic behavior.

In the *Eddington–Finkelstein coordinate system* a new variable v is introduced through

$$t = \underbrace{v}_{\text{new}} - r - 2M \ln \left| \frac{r}{2M} - 1 \right| ,$$

where r, t, and M have their usual meanings in the Schwarzschild metric, and θ and φ are assumed to be unchanged. For either $r > 2M$ or $r < 2M$, insertion into the standard Schwarzschild line element gives (Exercise)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 .$$

- The Schwarzschild metric expressed in these new coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

  is manifestly non-singular at $r = 2M$

- The singularity at $r = 0$ remains.
- Thus the singularity at the Schwarzschild radius is a *coordinate singularity* that can be removed by a new choice of coordinates.

Let us consider behavior of radial light rays expressed in these coordinates.

- Set $d\theta = d\varphi = 0$ (radial motion)
- Set $ds^2 = 0$ (light rays).

Then the Eddington–Finkelstein line element gives

$$-\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr = 0.$$

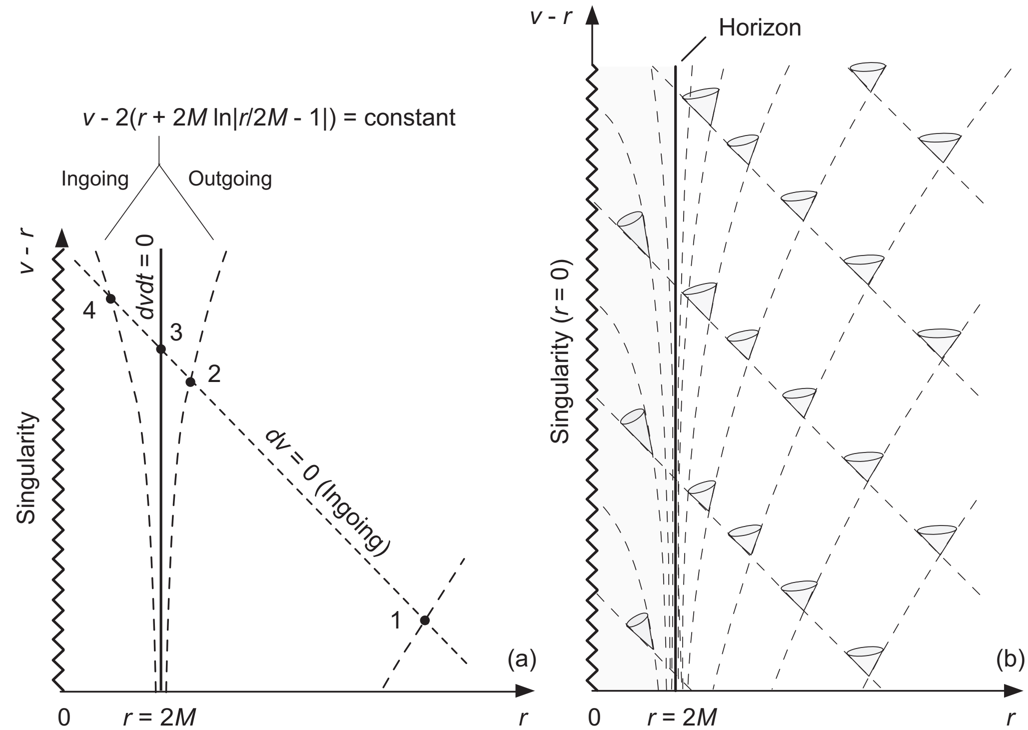
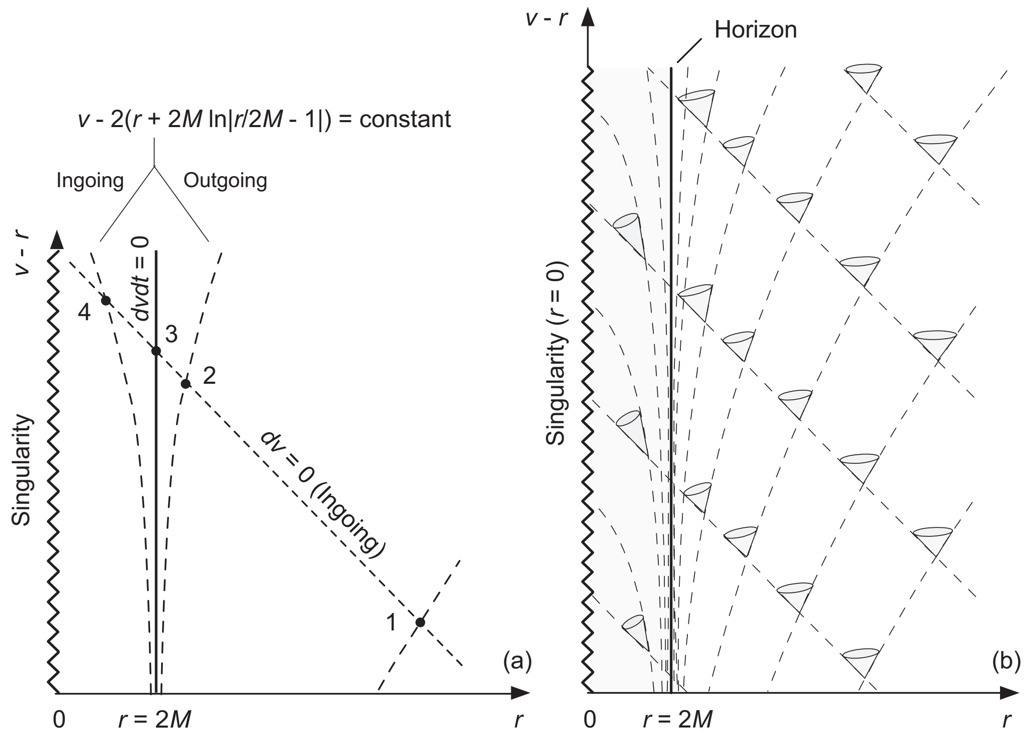


Figure 11.4: (a) Eddington–Finkelstein coordinates for the Schwarzschild black hole with r on the horizontal axis and $v-r$ on the vertical axis. Only two coordinates are plotted, so each point corresponds to a 2-sphere of angular coordinates. (b) Light cones in Eddington–Finkelstein coordinates.

This equation has two general solutions and one special solution [see Fig. 11.4(a)]:

$$-\left(1-\frac{2M}{r}\right)dv^2+2dvdr=0.$$

- *General Solution 1:* $dv=0$, so $v=$ constant. $\rightarrow$ Ingoing light rays on trajectories of constant v (dashed lines Fig. 11.4(a)) .



- *General Solution 2:* If $dv \neq 0$, then divide by dv^2 to give

$$-\left(1-\frac{2M}{r}\right)dv^2+2dvdr=0 \quad \rightarrow \quad \frac{dv}{dr}=2\left(1-\frac{2M}{r}\right)^{-1},$$

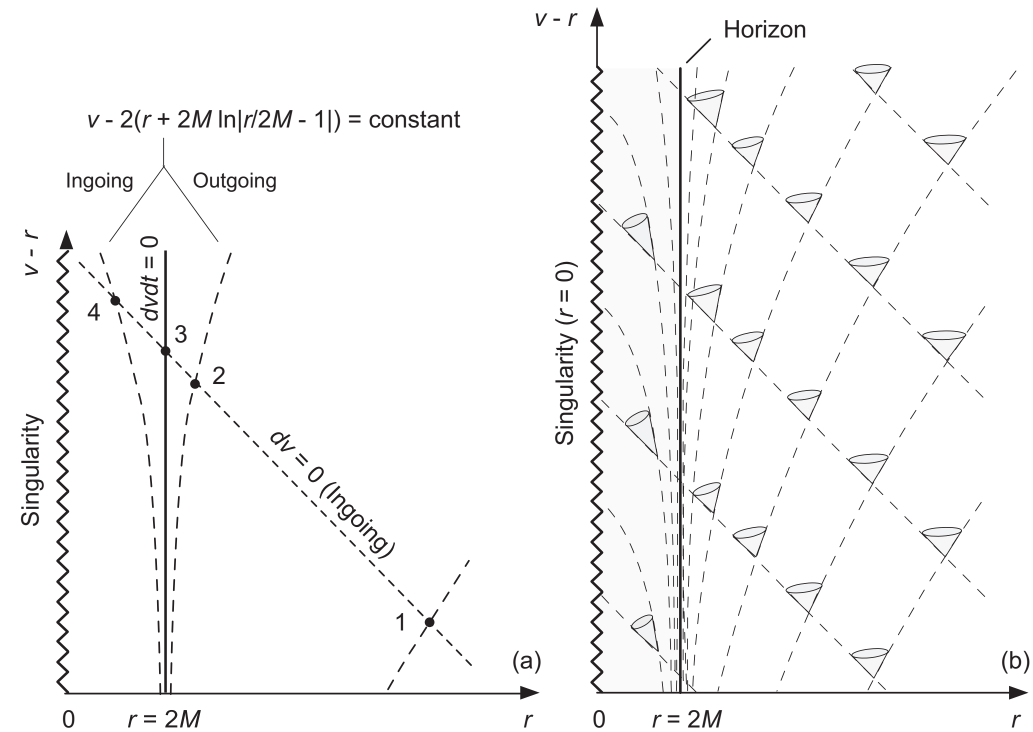
which yields upon integration

$$v-2\left(r+2M\ln\left|\frac{r}{2M}-1\right|\right) = \text{ constant.}$$

This solution changes behavior at $r = 2M$:

1. *Outgoing* for $r > 2M$.
2. *Ingoing* for $r < 2M$ (r decreases as v increases).

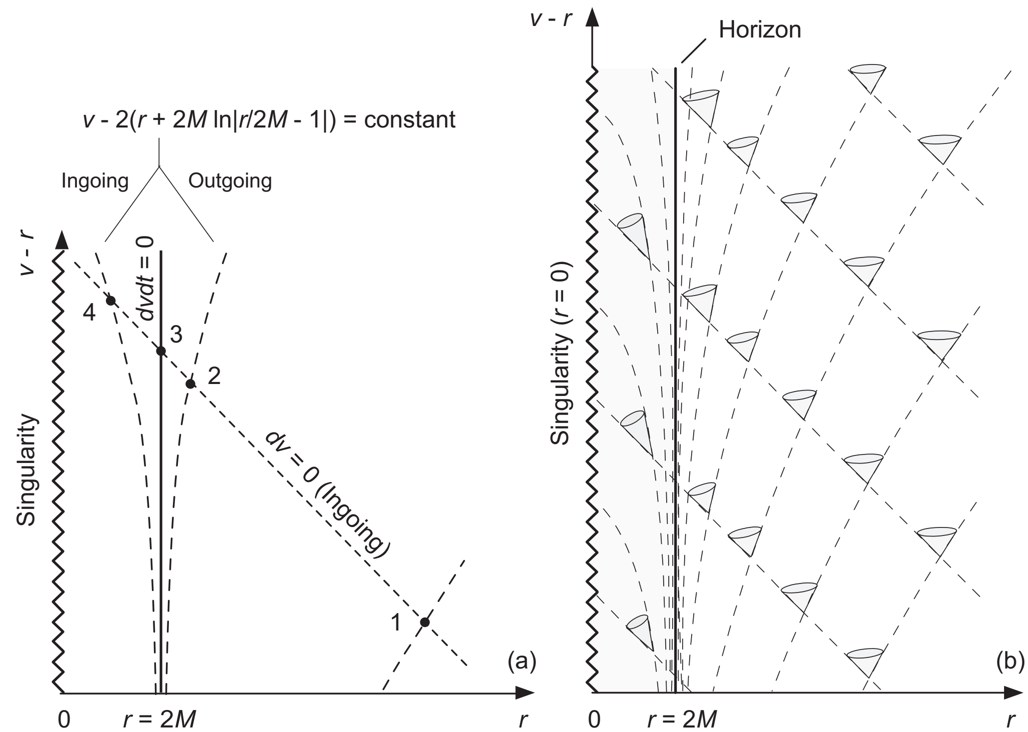
The long-dashed curves in Fig. 11.4(a) illustrate both ingoing and outgoing world-lines corresponding to this solution.



• *Special Solution:* In the special case that $r = 2M$, the differential equation reduces to

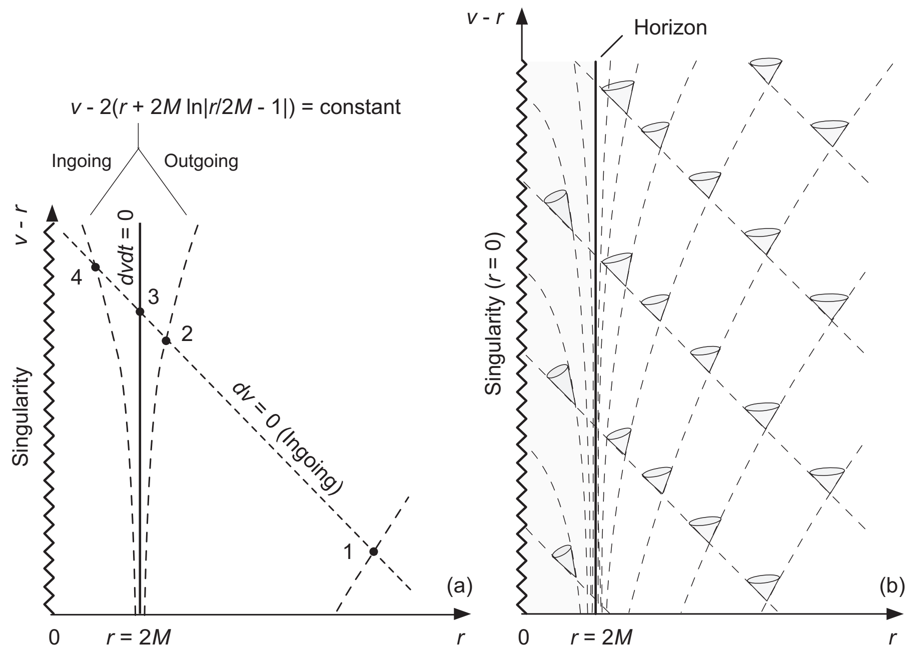
$$-\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr = 0 \quad \rightarrow \quad dvdr = 0,$$

which corresponds to light trapped at the Schwarzschild radius. The vertical solid line at $r = 2M$ represents this solution.



For every spacetime point in Fig. 11.4(a) there are two solutions.

- For the points labeled 1 and 2 these correspond to an ingoing and outgoing solution.
- For point 3 one solution is ingoing and one corresponds to light trapped at $r = r_S$.
- For point 4 *both solutions are ingoing*.



The two solutions passing through a point determine the light cone structure at that point (right side of figure).

- The light cones at various points are bounded by the two solutions, so they tilt "inward" as r decreases.
- The radial light ray that defines the left side of the light cone is ingoing (general solution 1).
- If $r \neq 2M$, the radial light ray defining the right side of the light cone corresponds to general solution 2.
  1. These propagate outward if $r > 2M$.
  2. For $r < 2M$ they propagate *inward*.
- For $r < 2M$ the light cone is tilted sufficiently that no light ray can escape the singularity at $r = 0$.
- At $r = 2M$, one light ray moves inward; one is trapped at $r = 2M$.

The horizon may be viewed as a null surface generated by the radial light rays that can neither escape to infinity nor fall in to the singularity.

11.1.5 Kruskal–Szekeres Coordinates

There is another set of coordinates exhibiting no singularity at $r = 2M$: *Kruskal–Szekeres coordinates.*

- Introduce variables (v, u, θ, φ), where θ and φ have their usual meaning and the new variables u and v are defined through

$$
\begin{aligned}
u &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \qquad (r > 2M) \\
&= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \qquad (r < 2M) \\
v &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \qquad (r > 2M) \\
&= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \qquad (r < 2M)
\end{aligned}
$$

- The corresponding line element is

$$
ds^2 = \frac{32M^3}{r} e^{-r/2M} (-dv^2 + du^2) + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2,
$$

  where $r = r(u, v)$ is defined through

$$
\left(\frac{r}{2M} - 1\right) e^{r/2M} = u^2 - v^2.
$$

- This metric is manifestly non-singular at $r = 2M$, but singular at $r = 0$.

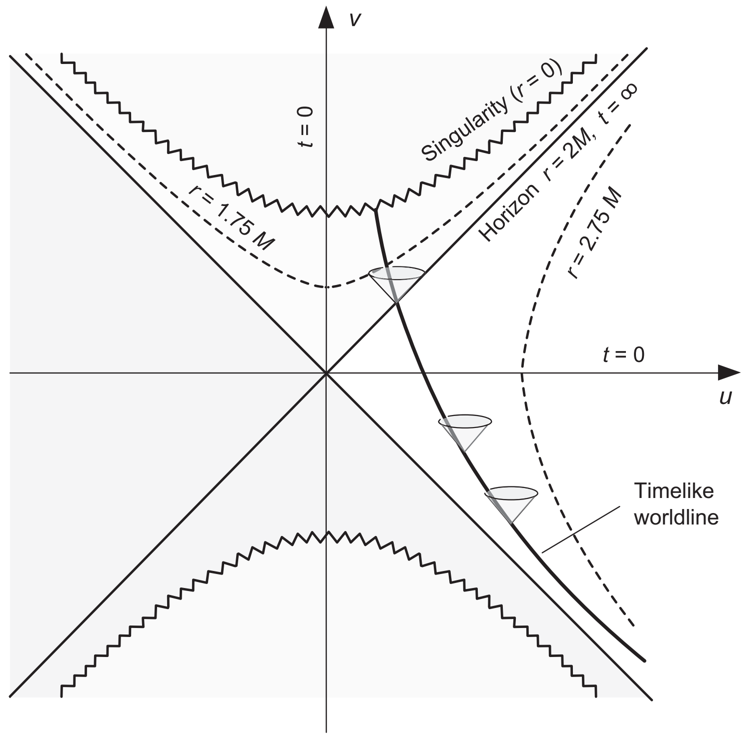
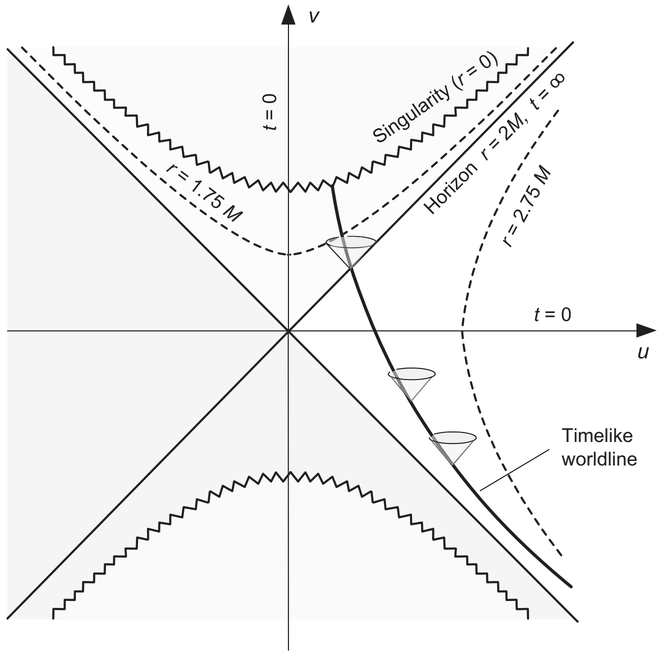


Figure 11.5: Kruskal–Szekeres coordinates. Only the two coordinates u and v are displayed; each point is really a 2-sphere in the variables θ and φ.

Kruskal diagram: lines of constant r and t plotted on a u and v grid. Figure 11.5 illustrates.



- From the form of

$$\left(\frac{r}{2M}-1\right)e^{r/2M}=u^2-v^2.$$

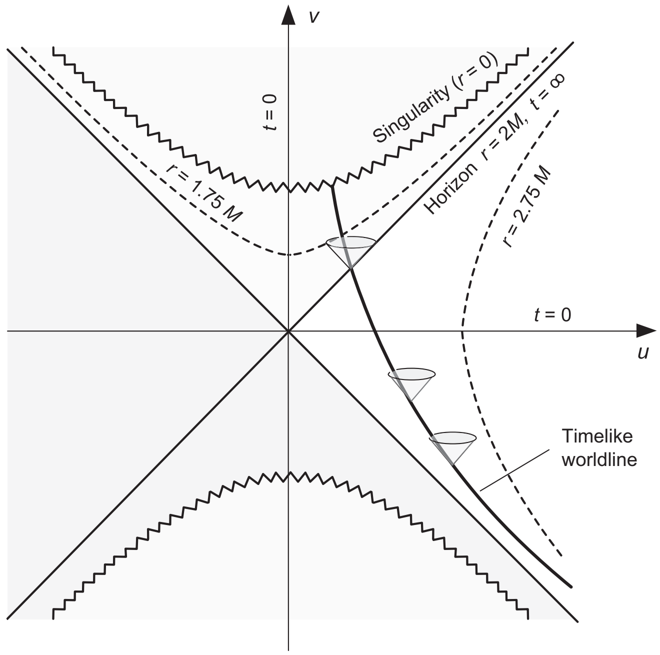
lines of constant r are hyperbolae of constant u^2-v^2.

- From the definitions of u and v

$$\begin{aligned} v &= u\tanh\left(\frac{t}{4M}\right) && (r>2M) \\ &= \frac{u}{\tanh(t/4M)} && (r<2M). \end{aligned}$$

Thus, lines of constant t are straight lines with slope

- $\tanh(t/4M)$ for $r>2M$
- $1/\tanh(t/4M)$ for $r<2M$.



- For radial light rays in Kruskal–Szekeres coordinates ($d\theta = d\varphi = ds^2 = 0$), and the line element

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dv^2 + du^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

yields $dv = \pm du$:

> 45 degree lightcones in the uv parameters, like flat space.

- Over the full range of Kruskal–Szekeres coordinates (v, u, θ, φ), the metric component $g_{00} = g_{vv}$ remains negative and $g_{11} = g_{uu}$, $g_{22} = g_{\theta\theta}$, and $g_{33} = g_{\varphi\varphi}$ remain positive.
- Therefore, the v direction is always timelike and the u direction is always spacelike, in contrast to the normal Schwarzschild coordinates where r and t switch their character at the horizon.

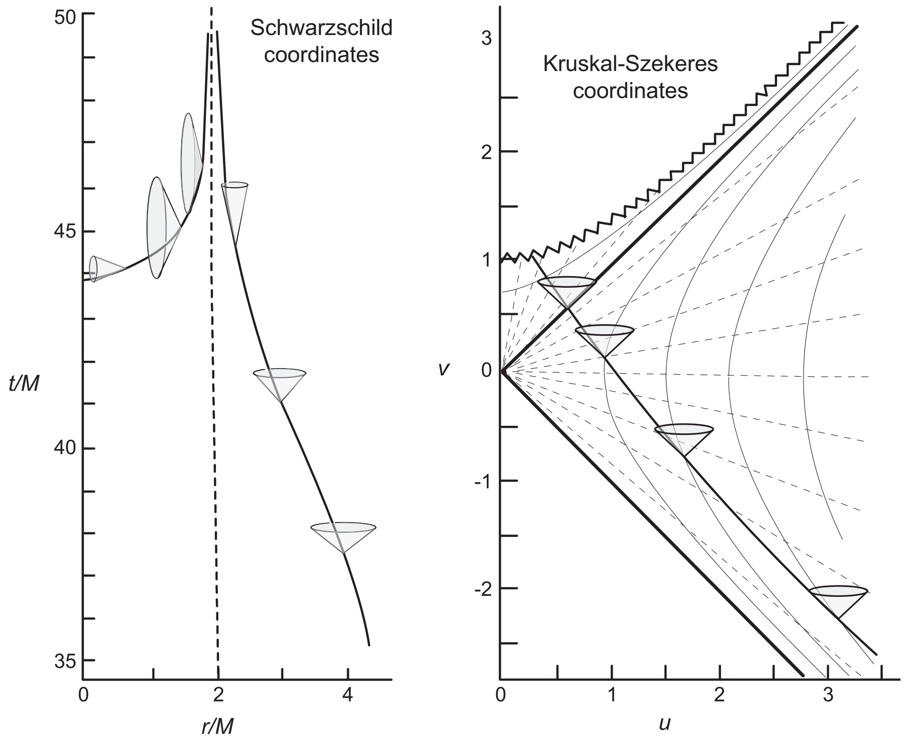


Figure 11.6: A trip to the center of a black hole in standard Schwarzschild coordinates and in Kruskal–Szekeres coordinates.

The identification of $r = 2M$ as an event horizon is particularly clear in Kruskal–Szekeres coordinates (Fig. 11.6).

- The light cones make 45-degree angles with the vertical and the horizon also makes a 45-degree angle with the vertical.
- Thus, for any point within the horizon, its forward worldline must contain the $r = 0$ singularity and cannot contain the $r = 2M$ horizon.

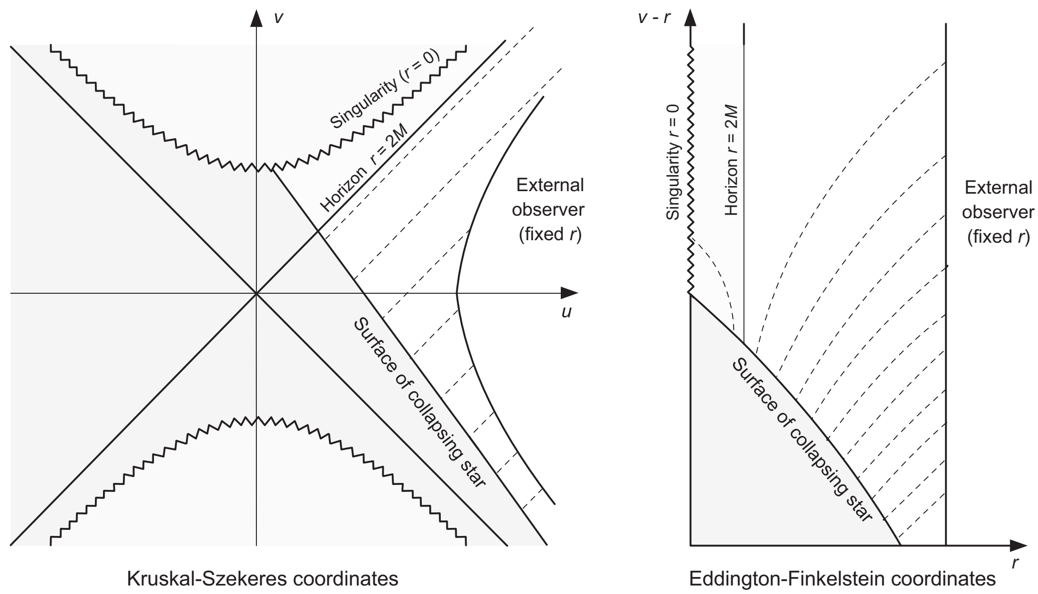


Figure 11.7: Collapse to a Schwarzschild black hole.

Figure 11.7 illustrates a spherical mass distribution (a star, for example) collapsing to a black hole as represented in Kruskal–Szekeres coordinates and in Eddington–Finkelstein coordinates.

- A distant observer remains at fixed r and observes light signals sent periodically from the surface of the collapsing star.
- Light pulses, propagating on the dashed lines, arrive at longer and longer intervals as measured by the outside observer.
- At the horizon, light signals take an infinite length of time to reach the external observer.
- Once the surface is inside the horizon, no signals can reach the outside observer and the entire star collapses to the singularity.
- *Note:* the Schwarzschild solution is valid only *outside* the star. Inside GR applies but the solution is not Schwarzschild.

11.2 Black Hole Theorems and Conjectures

In this section we summarize (in a non-rigorous way) a set of theorems and conjectures concerning black holes. Some we have already used in various contexts.

- Singularity theorems: Loosely, any gravitational collapse that proceeds far enough results in a spacetime singularity.
- Cosmic censorship conjecture: All spacetime singularities are hidden by event horizons (no naked singularities).
- (Classical) area increase theorem: In all classical processes involving horizons, the area of the horizons can never decrease.
- Second law of black hole thermodynamics: Where quantum mechanics is important the classical area increase theorem is replaced by

    1. The entropy of a black hole is proportional to the surface area of its horizon.
    2. The total entropy of the Universe can never decrease an any process.

- The no-hair theorem/conjecture: If gravitational collapse to a black hole is nearly spherical,
  - All non-spherical parts of the mass distribution (quadrupole moments, ...) except angular momentum are radiated away as gravitational waves.
  - Horizons eventually become stationary.
  - A stationary black hole is characterized by three numbers: the mass M, the angular momentum J, and the charge Q.
  - M, J, and Q are all determined by fields outside the horizon, not by integrals over the interior.

The most general solution characterized by M, J, and Q is termed a Kerr–Newman black hole. However,

- It is likely that the astrophysical processes that could form a black hole would neutralize any excess charge.
- Thus astrophysical black holes are Kerr black holes (the Schwarzschild solution being a special case of the Kerr solution for vanishing angular momentum).

The "No Hair Theorem": black holes destroy all details (the hair) about the matter that formed them, leaving behind only global mass, angular momentum, and possibly charge as observable external characteristics.

- Birkhoff's theorem: The Schwarzschild solution is the only spherically symmetric solution of the vacuum Einstein equations. (The static assumption is, in fact, a consequence of the spherical symmetry assumption.)

These theorems and conjectures place the mathematics of black holes on reasonably firm ground. To place the *physics* of black holes on firm grounds, these ideas must be tested by observation, which we shall take up in Ch. 15.